

CALCULATION OF THE NORM OF THE ERROR FUNCTIONAL OF OPTIMAL QUADRATURE FORMULAS IN THE SPACE $W_2^{(2,1)}(0, 1)$

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Abstract. In this paper in the space $W_2^{(2,1)}(0, 1)$ square of the norm of the error functional of a optimal quadrature formula is calculated.

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Many well known mathematicians be occupied with construction of optimal formulas for approximate integration. Full bibliography by this direction can be found in books [1,2,3,4].

In works [5,6] in space $W_2^{(m,m-1)}(0, 1)$ was considered problem of construction of the optimal quadrature formulas of the form

$$\int_0^1 p(x)\varphi(x)dx \cong \sum_{\beta=0}^N C_\beta \varphi(x_\beta) \quad (1)$$

with error functional

$$\ell(x) = p(x)\varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N C_\beta \delta(x - x_\beta), \quad (2)$$

where C_β and $x_\beta \in [0, 1]$ ($\beta = \overline{0, N}$) are called *coefficients* and *nodes* of the quadrature formula (1) respectively, $p(x)$ is a weight function, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval $[0,1]$, $\delta(x)$ is Dirac delta function, and $\varphi(x)$ is such a function, that is contained in Hilbert space $W_2^{(m,m-1)}(0, 1)$ norm of functions in which is defined by formula

$$\|\varphi(x)|W_2^{(m,m-1)}(0, 1)\| = \left\{ \int_0^1 (\varphi^{(m)}(x) + \varphi^{(m-1)}(x))^2 dx \right\}^{1/2}.$$

Note that in work [5] was found the extremal function and with its help following representation of square of the norm of the error functional (2) was obtained:

$$\begin{aligned} \|\ell(x)\|_{W_2^{(m,m-1)*}(0,1)}^2 = & (-1)^m \left[\sum_{\beta=0}^N \sum_{\beta'=0}^N C_\beta C_{\beta'} \psi_m(x_\beta - x_{\beta'}) - \right. \\ & \left. - 2 \sum_{\beta=0}^N C_\beta \int_0^1 p(x) \psi_m(x - x_\beta) dx + \int_0^1 \int_0^1 p(x) p(y) \psi_m(x - y) dx dy \right], \end{aligned} \quad (3)$$

where

$$\psi_m(x) = \frac{\text{sign} x}{2} \left(\frac{e^x - e^{-x}}{2} - \sum_{k=1}^{m-1} \frac{x^{2k-1}}{(2k-1)!} \right). \quad (4)$$

Moreover, the error functional (2), as shown in [5], satisfies following orthogonality conditions

$$(\ell(x), x^\alpha) = 0, \quad \alpha = \overline{0, m-2}, \quad (5)$$

$$(\ell(x), e^{-x}) = 0. \quad (6)$$

The norm (3) of the error functional $\ell(x)$ is many-dimensional function of the coefficients C_β ($\beta = \overline{0, N}$). Since error of the quadrature formula (1) is estimated from above by the norm of the error functional $\ell(x)$ in conjugate space, then in order to construct the optimal quadrature formula of the form (1) it is required to minimize, taking account of conditions (5) and (6), square of the norm (3) by coefficients C_β when the nodes x_β are fixed, i.e. we need find condition minimum of square of the error functional norm in conditions (5), (6).

Further in [5], applying the method of Lagrange undetermined factors, for finding of minimum of square of the error functional norm following discrete system of Wiener-Hopf type was obtained

$$\sum_{\gamma=0}^N C_\gamma \psi_m(x_\beta - x_\gamma) + P_{m-2}(x_\beta) + de^{-x_\beta} = \int_0^1 p(x) \psi_m(x - x_\beta) dx, \quad \beta = \overline{0, N}, \quad (7)$$

$$\sum_{\gamma=0}^N C_\gamma x_\gamma^\alpha = \int_0^1 p(x) x^\alpha dx, \quad \alpha = \overline{0, m-2}, \quad (8)$$

$$\sum_{\gamma=0}^N C_\gamma e^{-x_\gamma} = \int_0^1 p(x) e^{-x} dx, \quad (9)$$

where $\psi_m(x)$ is defined by formula (4), $P_{m-2}(x)$ is unknown polynomial of degree $m-2$, d is unknown constant. And also existence and uniqueness of the solution of this system were proved.

In [6] when $x_\beta = h\beta$, $\beta = \overline{0, N}$, $h = \frac{1}{N}$, $N = 1, 2, \dots$ the system (7)-(9) was solved and it was found the analytical representations of the coefficients C_β ($\beta = \overline{0, N}$). The coefficients for which minimum of square of the error functional norm is attained are called by *optimal*. In particular, for $m = 2$, $p(x) = 1$ following theorem was proved

Theorem 1. *The coefficients of optimal quadrature formulas of the form (1) in the space $W_2^{(2,1)}(0, 1)$ when $p(x) = 1$ have following view:*

$$C_\beta = \begin{cases} \frac{e^h - 1 - h}{e^h - 1} - K(h)(\lambda_1 - \lambda_1^N), & \beta = 0 \\ h - K(h) \left[(\lambda_1 - e^h) \lambda_1^\beta + (\lambda_1 e^h - 1) \lambda_1^{N-\beta} \right], & \beta = \overline{1, N-1}, \\ \frac{he^h - e^h + 1}{e^h - 1} - K(h)(\lambda_1 - \lambda_1^N) e^h, & \beta = N, \end{cases} \quad (10)$$

where

$$K(h) = \frac{(2e^h - 2 - he^h - h)(\lambda_1 - 1)}{2(e^h - 1)^2(\lambda_1 + \lambda_1^{N+1})}, \quad (11)$$

$$\lambda_1 = \frac{h(e^{2h} + 1) - e^{2h} + 1 - (e^h - 1)\sqrt{h^2(e^h + 1)^2 + 2h(1 - e^h)}}{1 - e^{2h} + 2he^h}. \quad (12)$$

Present paper is direct continuation of works [5,6]. Aim of given work is with using theorem 1 to calculate the square of the norm of the error functional $\ell(x)$ in the space $W_2^{(2,1)*}(0, 1)$.

Following is take placed

Теорема 2. For square of the norm (3) of the functional (2) of optimal quadrature formula (1) when $p(x) = 1$, $x_\beta = h\beta$ in the space $W_2^{(2,1)*}(0, 1)$ following equality is valid

$$\begin{aligned} \|\ell(x)\|_{W_2^{(2,1)*}(0,1)}^2 &= \frac{h^2}{12} + \frac{h(2 - e^h - 3e^{2h}) + 4 + 2e^h + 6e^{2h}}{4(1 - e^h)^2} + \\ &+ K(h) \left[\frac{(\lambda_1^N + \lambda_1^2)(1 + e^h) - (\lambda_1^{N+1} + \lambda_1)(1 + 2e^h)}{2(1 - \lambda_1)} + \right. \\ &\quad + \frac{h^2(\lambda_1^2 + \lambda_1)(\lambda_1^N - 1)(1 + e^h)}{2(1 - \lambda_1)^2} + \\ &\quad \left. + \frac{(\lambda_1 - e^h)^2(\lambda_1^N - \lambda_1 e^h) - (1 - \lambda_1 e^h)^2(\lambda_1 - \lambda_1^N e^h)}{2(1 - \lambda_1 e^h)(\lambda_1 - e^h)} \right], \end{aligned}$$

where $K(h)$ and λ_1 are determined by (11) and (12) respectively.

Proof. The system (7)-(9) for $m = 2$, $p(x) = 1$, $x_\beta = h\beta$ have form

$$\sum_{\gamma=0}^N C_\gamma \psi_2(h\beta - h\gamma) + P_0(h\beta) + d e^{-h\beta} = \int_0^1 \psi_2(x - h\beta) dx, \quad (13)$$

$$\sum_{\gamma=0}^N C_\gamma = 1, \quad (14)$$

$$\sum_{\gamma=0}^N C_\gamma e^{-h\gamma} = 1 - e^{-1}. \quad (15)$$

Then for square of the norm (3) of the functional $\ell(x)$ we obtain

$$\begin{aligned} \|\ell(x)\|^2 &= \sum_{\beta=0}^N C_\beta \left(\sum_{\beta'=0}^N C_{\beta'} \psi_2(h\beta - h\beta') - \int_0^1 \psi_2(x - h\beta) dx \right) - \\ &- \sum_{\beta=0}^N C_\beta \int_0^1 \psi_2(x - h\beta) dx + \int_0^1 \int_0^1 \psi_2(x - y) dx dy. \end{aligned}$$

Hence, taking into account (13), we have

$$\|\ell(x)\|^2 = - \sum_{\beta=0}^N C_\beta (P_0(h\beta) + d e^{-h\beta}) -$$

$$-\sum_{\beta=0}^N C_{\beta} \int_0^1 \psi_2(x - h\beta) dx + \int_0^1 \int_0^1 \psi_2(x - y) dx dy. \quad (16)$$

From here, taking account of (4), for integrals of (16) we get

$$\int_0^1 \psi_2(x - h\beta) dx = \frac{e^{h\beta} + e^{-h\beta} + e^{1-h\beta} + e^{h\beta-1} - 4}{4} - \frac{(h\beta)^2 + (1 - h\beta)^2}{4}, \quad (17)$$

$$\int_0^1 \int_0^1 \psi_2(x - y) dx dy = \frac{e^2 - 1}{2e} - \frac{7}{6}. \quad (18)$$

In representation (16) of the error functional norm polynomial $P_0(h\beta) = b_0$ and constant d are unknowns. For $\|\ell(x)\|^2$, using (14), (15), (17), (18), we have

$$\begin{aligned} \|\ell(x)\|^2 = & -b_0 + \frac{1-e}{e}d - \frac{e+1}{4e} \sum_{\beta=0}^N C_{\beta} e^{h\beta} - \frac{1+e}{4}(1 - e^{-1}) + \frac{5}{4} + \\ & + \frac{1}{2} \sum_{\beta=0}^N C_{\beta} (h\beta)^2 - \frac{1}{2} \sum_{\beta=0}^N C_{\beta} (h\beta) + \frac{e^2 - 1}{2e} - \frac{7}{6}. \end{aligned} \quad (19)$$

The equality (13) take placed in any $h\beta$ when $\beta = \overline{0, N}$, $h = \frac{1}{N}$, i.e. is identity by powers $h\beta$ and $e^{h\beta}$, $e^{-h\beta}$. Equating coefficients of the left and the right sides of (13) in front of $e^{-h\beta}$ and constant term, by using theorem 1, equalities (4), (14), (15), (17), for d and b_0 we get

$$\begin{aligned} d = & \frac{C_0}{2} + \frac{1}{2} \left(\frac{he^h}{1 - e^h} + a_1 \frac{\lambda_1 e^h}{1 - \lambda_1 e^h} + b_1 \frac{\lambda_1^N e^h}{\lambda_1 - e^h} \right) - \frac{1}{4} \sum_{\gamma=0}^N C_{\gamma} e^{h\gamma} + \frac{1+e}{4}, \\ -b_0 = & \frac{h(1 + e^h)}{2(1 - e^h)} + ha_1 \frac{\lambda_1}{(1 - \lambda_1)^2} + hb_1 \frac{\lambda_1^{N+1}}{(1 - \lambda_1)^2} - \frac{1}{2} \sum_{\gamma=0}^N C_{\gamma} (h\gamma) + \frac{5}{4}, \end{aligned}$$

where $a_1 = K(h)(e^h - \lambda_1)$, $b_1 = K(h)(1 - \lambda_1 e^h)$.

Then from (19), taking account of d and b_0 , using identities

$$\begin{aligned} \sum_{\gamma=1}^{N-1} \lambda^{\gamma} \gamma &= \frac{\lambda - \lambda^{N+1} - N\lambda^N(1 - \lambda)}{(1 - \lambda)^2}, \\ \sum_{\gamma=1}^{N-1} \lambda^{\gamma} \gamma^2 &= \frac{\lambda^N(\lambda^2 + \lambda + N^2(1 - \lambda)^2 + 2N(\lambda - \lambda^2))}{(\lambda - 1)^3} - \frac{\lambda^2 + \lambda}{(\lambda - 1)^3} \end{aligned}$$

and theorem 1, after simplification we get the statement of the theorem 2.

References

1. Krylov V.I. Approximate Calculation of Integrals. - M.: Nauka, 1967. - 500p.
2. Nikolskii S.M. Quadrature Formulas. - M.: Nauka, 1988. - 256 p.
3. Sobolev S.L. Introduction to the Theory of Cubature Formulas. -M.: Nauka, 1974. - 808p.
4. Sobolev S.L., Vaskevich V.L. Cubature Formulas. -Novosibirsk, Institute of Mathematics SB of RAS, 1996, -484p.
5. Shadimetov Kh.M., Hayotov A.R. Weight Optimal Quadrature Formulas in the Space $W_2^{(m,m-1)}(0,1)$. Uzbek Mathematical Journal. 2002. №3-4, pp.92-103.
6. Shadimetov Kh.M., Hayotov A.R. Calculation of the Coefficients of Optimal Quadrature Formulas in the $W_2^{(m,m-1)}(0,1)$. Uzbek Mathematical Journal. 2004. №3. pp.67-82.
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